stichting
mathematisch
centrum

MC

AFDELING ZUIVERE WISKUNDE ZW 42/75 DECEMBER

J. DE VRIES

CATEGORIES OF TOPOLOGICAL TRANSFORMATION GROUPS

Prepublication

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): Primary: 54H15, 18B99

Secondary: 18C15, 43A15

Categories of topological transformation groups *)

by

J. de Vries

ABSTRACT

In the first part of this paper a number of questions is considered concerning some particular categories of topological transformation groups (ttg's). In addition to results which are interesting from the point of view of category theory, some important "classical" problems and techniques are mentioned which appear in a natural way from simple categorical considerations. The second part of this paper is connected with the particular problem of embedding ttg's in certain systems consisting of groups of linear operators acting on topological vector spaces. The solutions of this problem can be given without categorical methods. However, in placing the problem and its solutions into a categorical setting, some interesting categories naturally arise.

KEY WORDS & PHRASES: *topological transformation group, category theory, equivariant embedding, groups of linear operators.*

---

*) This paper is not for review; it is meant to be published elsewhere.

1. INTRODUCTION

The theory of topological transformation groups (ttg's) forms a fascinating and comprehensive realm in the world of mathematics, bordering on the domains of abstract harmonic analysis, ergodic theory, geometry, differential equations and topology. In this talk I cannot give you even a flavour of the subject. Instead, I would like to discuss certain *categories of ttg's*. I shall use category theory in a rather "naive" way. Some categories of ttg's will be defined and investigated. In this context, the categories TOPGRP (all topological groups and continuous homomorphisms) and TOP (all topological spaces and continuous functions) will be regarded as "known", and most questions will be reduced to questions about these categories. In the attempt to do so, some "classical" problems and techniques appear in a natural way. This shows that these problems are interesting, not only because they turned out to be so in the development of the subject (by "accident"), but also from the more "intrinsic" point of view of category theory. However, the solutions of these problems have been given independently of category theory. On the other hand, attempts to place certain problems and their solutions into a categorical setting can be very illuminating, and may require the definition of interesting new categories.
This will be illustrated when we consider the problem of embedding arbitrary ttg's in linear ttg's.

Let me first recall some definitions. A *topological transformation group* (ttg) is a system $\langle G,X,\pi\rangle$ in which G is a topological group (the *phase group*), X is a topological space (the *phase space*) and π (the *action* of G on X) is a continuous function, $\pi\colon G\times X\to X$, such that

$$\pi(e,x) = x;\ \pi(s,\pi(t,x)) = \pi(st,x)$$

for every $x \in X$ and $s, t \in G$ (e denotes the unit of any group under consideration). A ttg with phase group G is often called *a G-space*. If π is an action of G on X, continuous mappings $\pi^t: X \to X$ and $\pi_x: G \to X$ can be defined by

$$\pi^t x := \pi(t,x) =: \pi_x t$$

for $t \in G$ and $x \in X$. Plainly, each π^t is an autohomeomorphism of X, and the mapping $\bar{\pi}: t \mapsto \pi^t$ is a homomorphism of the underlying group of G into the full homeomorphism group $H(X)$ of X. The closure of the group $\bar{\pi}[G]$ in X^X is a semigroup (with composition of mappings as multiplication), called the *enveloping semigroup* of $\langle G,X,\pi\rangle$. For $x \in X$, the subset $\pi_x[G]$ of X is called the *orbit* of x under G. The orbits form a partition of X. The corresponding quotient space and quotient mapping are denoted by X/C_π and $c_\pi: X \to X/C_\pi$, respectively.

We give here a few examples of ttg's . In all cases, G denotes a topological group, and $\lambda: G \times G \to G$ its multiplication.

(i) For every topological space X, define $\mu_X^G := \lambda \times 1_X: G \times G \times X \to G \times X$. Then $\langle G, G\times X, \mu_X^G\rangle$ is a ttg.

(ii) If H is a subgroup of G then G acts on the space G/H of left cosets by means of an action π, defined by $\pi^t(sH) := tsH$, $s,t \in G$.

(iii) If Y is a topological space then a mapping $\tilde{\rho}: G \times C_c(G,Y) \to C_c(G,Y)$ can be defined by $\tilde{\rho}^t f(s) := f(st)$, $s,t \in G$, $f \in C(G,Y)$. If G is locally compact, then $\tilde{\rho}$ is continuous, and $\langle G, C_c(G,Y), \tilde{\rho}\rangle$ is a ttg.

(v) Similarly, a ttg $\langle G, L^p(G), \tilde{\rho}\rangle$ can be defined if G is locally compact and $1 \le p < \infty$. Here $L^p(G)$ is the usual space of measurable functions whose p-th power is integrable with respect to the right Haar measure on G.

(vi) If f is a sufficiently nice $\mathbb{R}^n$-valued function on an open domain Ω in $\mathbb{R}^n$, then the autonomous differential equation $\dot{x} = f(x)$ defines an action of $\mathbb{R}$ on Ω such that the orbits are just the solution curves. (Due to this example, actions of $\mathbb{R}$ on arbitrary spaces are often called *flows*.)

We cannot go into details here about the application of ttg's. Originating from differential geometry (the work of S. LIE on "continuous groups") there is the theory of Lie groups and their actions, including work of HILBERT, BROUWER, CARTAN and WEYL, to mention only a few names from the classical period. Introductions into these areas can be found in [29] or [9]. For applications in harmonic analysis, see for instance [40]. Related is the theory of fibre bundles.

Ttg's are also studied in Topological Dynamics. This field of research grew from classical dynamics and the qualitative theory of differential equations in an attempt to prove theorems about stability, recurrence, asymptoticity, etc. by purely topological means, whenever possible. The most notable early work was by H. POINCARÉ and G.D. BIRKHOFF. A large body of results for flows which are of interest for classical dynamics has been developed since that time, without reference to the fact that the flows arise from differential equations. Later, results were extended to general ttg's. A landmark in this development towards abstraction is the book [19]; a more recent introduction is [16]. Here the link between ttg's and dynamics is not so clear for a non-specialist. More closely related to differential equations are books like [8] or [21]. Also in some books on differential equations one can find results on flows. See for instance [31] or [25]*). These theories are "local", in the sense that questions are asked like "what does the ω-limit set look like?"; "what happens in the neighbourhood of a fixed point?"; etc. Related are the "global" theories of SMALE and others, where the object of study is vector fields on manifolds. For an introduction, see [2], and for applications, [1].

In the development of the theory of ttg's an important role has been played by the quotient mapping $c_\pi : X \to X/C_\pi$ for a ttg $\langle G,X,\pi \rangle$. Let me formulate here two related problems:

(i) If G is compact, then c_π is a perfect mapping, and there exists a nice relationship between the topological properties of X and X/C_π. In this context, also the normalized Haar measure of G can be used.

*) For flows in the plane, see also [7].

For which ttg's with a non-compact phase group does there exist such a nice relationship? Paracompactness of X/C_π turns out to be of particular interest.

(ii) A (global) *continuous cross-section* of a ttg $\langle G,X,\pi\rangle$ is a pair (S,τ) with $S \subseteq X$ and $\tau\colon X \to G$ a continuous function such that, for every $x \in X$, $\tau(x)$ is the *unique* element of G for which $\pi(\tau(x),x) \in S$. It is easily seen that $\langle G,X,\pi\rangle$ has a continuous cross-section iff it is isomorphic as a G-space with $\langle G,G\times Y,\mu_Y^G\rangle$ for some space Y. In that case, Y, S and X/C_π are homeomorphic. The question of which ttg's have such a global continuous cross-section is important, not only in abstract theories, but also in the study of flows ("which flows are *parallelizable?*").

These two problems are also related to general questions in Topological Dynamics, dealing with the structure of orbit closures. See e.g. [22]. Concerning the relationship between (i) and (ii), we confine ourselves to the remark that in order to prove that certain ttg's have a global continuous cross-section one usually shows first the existence of local continuous cross-sections; then, using nice properties of X/C_π and G, these are pasted together to a global one. Cf. for instance [34], [27], [22] and the references given there.

A number of solutions of the following problem use also the existence of local cross-sections for certain ttg's. The problem is

(iii) Which ttg's $\langle G,X,\pi\rangle$ can be embedded in a topological vector space V, or even in a Hilbert space, in such a way that the π^t's become restrictions of invertible linear operators ρ^t, $t \in G$, such that $\langle G,V,\rho\rangle$ is a ttg.

For flows, see for example (the proof of) BEBUTOV's theorem and generalizations thereof in [30], [26] or [23]. For actions of Lie groups G and embeddings in Hilbert G-spaces using the method of local cross-sections (or, more general, of *slices*), see [33] and the references given there. We shall return to problem (iii) in the last part of this paper. First, I shall indicate why problems (i) and (ii) are also interesting from a categorical point of view.

2. THE CATEGORY TTG

2.1. A *morphism of ttg's* from $\langle G,X,\pi\rangle$ to $\langle H,Y,\sigma\rangle$ is a morphism $(\psi,f)\colon (G,X) \to (H,Y)$ in the category TOPGRP × TOP for which the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\ \pi\ } & X \\ {\scriptstyle \psi\times f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ H \times Y & \xrightarrow[\ \sigma\ ]{} & Y \end{array}$$

commutes.[*] *Notation*: $\langle\psi,f\rangle\colon \langle G,X,\pi\rangle \to \langle H,Y,\sigma\rangle$. Here, f will be called a *ψ-equivariant* mapping; an 1_G-equivariant mapping will just be called equivariant.

Let TTG denote the category having the class of all ttg's as its object class (also ttg's with an empty phase space are admitted). The morphisms in TTG are the above defined morphisms of ttg's, with coordinatewise composition.

2.2. Important for the investigation of the category TTG are the following forgetful functors, whose obvious definitions we leave to the reader:

$$K\colon \text{TTG} \longrightarrow \text{TOPGRP} \times \text{TOP};$$
$$G\colon \text{TTG} \longrightarrow \text{TOPGRP};$$
$$S\colon \text{TTG} \longrightarrow \text{TOP}.$$

These functors forget all about actions, so they cannot be expected to reveal much about the "internal" structure of ttg's. In this respect, the following functor may be expected to be more useful:

$$S_1\colon \text{TTG} \longrightarrow \text{TOP}.$$

It is defined in the following way. For an object $\langle G,X,\pi\rangle$ in TTG, set $S_1\langle G,X,\pi\rangle := X/C_\pi$, the orbit space of $\langle G,X,\pi\rangle$. If $\langle\psi,f\rangle\colon \langle G,X,\pi\rangle \to \langle H,Y,\sigma\rangle$

[*] The products here are ordinary cartesian products, i.e. products in the category TOP. In this context, we shall consider TOPGRP just as a subcategory of TOP, and we shall always suppress the corresponding inclusion functor.

is a morphism in TTG, then f maps each orbit of X *into* an orbit of Y, hence there is a unique continuous function $f'\colon X/C_\pi \to Y/C_\sigma$ such that $f' \circ c_\pi = c_\sigma \circ f$. Now set $S_1\langle\psi,f\rangle := f'$.

2.3. THEOREM. *The functor* K: TTG $\to$ TOPGRP $\times$ TOP *is monadic. Consequently,* TTG *is complete, and* K *preserves and reflects all limits and all monomorphisms.*

PROOF. Let C: = TOPGRP $\times$ TOP, and define a functor $H\colon C \to C$ by means of the assignments

$$H\colon \begin{cases} (G,X) \mapsto (G,G\times X) & \text{on objects;} \\ (\psi,f) \mapsto (\psi,\psi\times f) & \text{on morphisms.} \end{cases}$$

Some straightforward arguments show that by

$$\eta_{(G,X)} := (1_G,\eta_X^G) \text{ and } \mu_{(G,X)} := (1_G,\mu_X^G),$$

(G,X) any object in C, two natural transformations

$$\eta\colon I_C \to H \quad\text{and}\quad \mu\colon H^2 \to H$$

are defined. Here $\eta_X^G(x) := (e,x)$ and $\mu_X^G(s,(t,x)) := (st,x)$ for $s,\ t \in G$ and $x \in X$. It is easily verified that the triple (H,η,μ) satisfies the definition of a monad (cf.[28], Chap.VI). The *algebras* over this monad are easily seen to be the systems $((G,X),\ (\psi,\pi))$ with (G,X) an object in C, $\psi = 1_G$, and $\pi\colon G \times X \to X$ a morphism in TOP making the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\eta_X^G} & G\times X \\ & {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle \pi} \\ & & X \end{array} \qquad\qquad \begin{array}{ccc} G\times(G\times X) & \xrightarrow{1_G\times\pi} & G\times X \\ {\scriptstyle \mu_X^G}\downarrow & & \downarrow{\scriptstyle \pi} \\ G\times X & \xrightarrow[\pi]{} & X \end{array}$$

commutative, i.e. π is an action of G on X. So the algebras over (H,η,μ) can unambiguously be identified with objects in TTG. In doing so, the morphisms between such algebras become morphisms in TTG, and the category of all algebras over (H,η,μ) turns out to be isomorphic (can be identified with) TTG. In making this identification, K corresponds to the forgetful functor of this category of algebras to C; this is equivalent to saying that K is monadic.

Now the remaining statements in the theorem are a direct consequence of the general theory of monads (cf. [28], Chap. VI). □

2.4. COROLLARY. *The functor* $K: TTG \to TOPGRP \times TOP$ *has a left adjoint* F, *defined by the rules*

$$F: \begin{cases} (G,X) \mapsto \langle G, G\times X, \mu_X^G\rangle & \text{on objects;} \\ (\psi,f) \mapsto \langle \psi, \psi\times f\rangle & \text{on morphisms.} \end{cases}$$

PROOF. Either by the theory of monads, using the identification of TTG with the category of algebras over (H,η,μ) as indicated in the proof of 2.3, or by a direct argument, showing that for every object (G,X) in TOPGRP × TOP the arrow

$$\eta_{(G,X)} : (G,X) \longrightarrow (G,G\times X)$$

has the desired universal property. □

2.5. The unit of the adjunction of F and K is the natural transformation η (cf. also the proof of 2.4); the counit is given by the arrows

$$\xi_{\langle G,X,\pi\rangle} : \langle G, G\times X, \mu_X^G\rangle \xrightarrow{\langle 1_G,\pi\rangle} \langle G,X,\pi\rangle$$

in TTG. Therefore, we may call the objects $\langle G, G\times X, \mu_X^G\rangle$ in TTG *free ttg's* (compare [24], p.231). This terminology can cause some confusion, because usually a ttg $\langle G,X,\pi\rangle$ is called free if $\pi^t x = x$ for some $x \in X$ implies $t = e$; we shall use the term *strongly effective* for this notion. It is obvious that a free ttg is strongly effective, but the converse is not generally true; a well-known class of counterexamples is provided by groups G which are subgroups of topological groups X such that the quotient mapping of H onto the space of right cosets does not admit a continuous section (let G act on X by left translations). *The free ttg's are plainly just the ttg's which have a continuous global cross-section.*

There is yet another way in which we arrive at the need of characterizing ttg's with continuous global cross-sections. Indeed, as in any adjoint situation, we have not only a monad, but also a comonad which is defined by the adjunction (F,K,η,ξ). Thus, in TTG, we have the comonad $(FK,\xi,F\eta K)$.

2.6. THEOREM. *The coalgebras for the comonad* $(FK,\xi,F\eta K)$ *in* TTG *are the systems* $(\langle G,X,\pi\rangle,(S,u))$ *with* (S,u) *a continuous cross-section of* $\langle G,X,\pi\rangle$. *The morphisms of coalgebras from* $(\langle G,X,\pi\rangle,(S,u))$ *to* $(\langle H,Y,\sigma\rangle,(T,v))$ *are the morphisms* $\langle\psi,f\rangle: \langle G,X,\pi\rangle \to \langle H,Y,\sigma\rangle$ in TTG *with* $f[S] \subseteq T$.

PROOF. Straightforward. For details, cf. [39], p. 92. □

2.7. As is well-known, comonads give rise to a cohomology theory (see e.g. [28], Chap.VII,§6). It would be interesting to investigate how this can be used with respect to the above mentioned comonad, (if not in TTG, then restricted to a suitable subcategory).

2.8. We shall show now that the category TTG is cocomplete, but that the functor K does not have nice preservation properties for colimits. In view of Beck's theorem (cf.[28], p.147) and theorem 2.3 above, certain coequalizers are preserved by K. An example where K does *not* preserve the coequalizer will be given in 2.13 below. The bad behaviour of K with respect to colimits is due to the functor S. Indeed, we have the following results:

2.9. LEMMA. *The functor* G: TTG $\to$ TOPGRP *has a right adjoint. Hence it preserves all colimits and all epimorphisms.* □

2.10. LEMMA. *The functor* S_1: TTG $\to$ TOP *has a right adjoint. Hence it preserves all colimits and all epimorphisms.* □

2.11. COROLLARY. *The functor* K: TTG $\to$ TOPGRP $\times$ TOP *preserves and reflects all epimorphisms.*

PROOF. Reflection: K is faithful.
Preservation: If <ψ,f>: <G,X,π> → <H,Y,σ> is epic in TTG, then by 2.9, ψ is a surjection. Hence f maps each orbit of X *onto* some orbit in Y. By 2.10, $S_1 f$ maps X/C_π onto Y/C_σ. Combining these results it follows easily that f is a surjection. □

2.12. The category TOPGRP × TOP is well-powered and co-(well-powered). Since the functor K preserves all monomorphisms and all epimorphisms, *the category* TTG *is well-powered and co-(well-powered)* as well. In addition, TTG is complete. Hence theorem 23.11 in [24] implies that *the category* TTG *has all coequalizers.*

2.13. EXAMPLE. Let <G,Y,σ> be a ttg. Call an equivalence relation X in Y *invariant* if, considered as a subset of Y × Y, it is invariant under the coordinate-wise action π of G on Y × Y. In order to obtain an example which shows that the functor K (or rather, the functor S) does not preserve co-equalizers, it is sufficient to construct a ttg <G,Y,σ> and an invariant equivalence relation X in Y such that there exists *no* continuous action of G on Y/X making the quotient mapping g: Y → Y/X equivariant.

To this end, take G: = ℚ, the additive group of the rationals, Y: = ℚ × ([0,1] × ℕ), and X: = $\Delta_{\mathbb{Q}}$ × R, where $\Delta_{\mathbb{Q}}$ is the diagonal in ℚ × ℚ and R is the equivalence relation in [0,1] × ℕ obtained by identifying all points (0,n), n ∈ ℕ, with each other. If we consider the action $\mu^{\mathbb{Q}}_{[0,1]\times\mathbb{N}}$ of ℚ on Y = ℚ × ([0,1] × ℕ), then the equivalence relation X in Y is invariant, and there is *only one candidate* ζ for an action of ℚ on Y/X which makes the quotient mapping g: Y → Y/X equivariant. Now continuity of ζ: ℚ × (Y/X) → Y/X can easily be seen to imply the equality of the following two topologies on Y/X: (i) the quotient topology induced by g and (ii) the product topology obtained by identifying Y/X with ℚ × (([0,1] × ℕ)/R), where ([0,1]×ℕ)/R has its usual quotient topology. It can be seen, however, that these two topologies do not coincide. For details, cf. [39], 3.4.4.

2.14. The easiest way to prove that TTG is cocomplete is by invoking theorem 23.13 in [24]. First, we recall some definitions. An *epi-sink* in a category X is a family $\{f_i: X_i \to X\}_{i\in I}$ of morphisms in X such that for every pair of morphisms g, h: $X \to Y$ in X the condition $gf_i = hf_i$ for all $i \in I$ implies $g = h$. The category X is called *strongly co-(well-powered)* if for every set-indexed family $\{X_i : i \in I\}$ of objects there is at most a set of objects X in X for which there exists an epi-sink $\{f_i: X_i \to X\}_{i\in I}$. The theorem referred to above reads as follows:

If the category X *is complete and well-powered, then the following are equivalent:*

(i) X *is strongly co-(well-powered);*

(ii) X *is cocomplete and co-(well-powered).*

Observe that the categories TOPGRP and TOP are complete and well-powered, and that they satisfy condition (ii). Hence these categories are strongly co-(well-powered). We shall use this in proving the following

2.15. LEMMA. *The category* TTG *is strongly co-(well-powered).*

PROOF. Let $\{\langle\psi_i,f_i\rangle: \langle G_i,X_i,\pi_i\rangle \to \langle G,X,\pi\rangle\}_{i\in I}$ be a set-indexed epi-sink in TTG. Since left adjoints preserve epi-sinks, if follows from 2.9 that $\{\psi_i: G_i \to G\}_{i\in I}$ is an epi-sink in TOPGRP. This allows G only to be taken from a set of possible topological groups. Similarly, 2.10 implies that $\{S_1f_i: X_i/C_{\pi_i} \to X/C_\pi\}_{i\in I}$ is an epi-sink in TOP, leaving for X/C_π only a set of possibilities. Plainly, $\mathrm{card}(X) \le \mathrm{card}(G).\mathrm{card}\ (X/C_\pi)$, hence there is at most a set of possibilities for X. Finally, for each G and each X there is only a set of actions of G on X. So there is at most a set of objects in TTG from which $\langle G,X,\pi\rangle$ can be taken. □

2.16. THEOREM. *The category* TTG *is cocomplete.*

PROOF. Clear from 2.14 and 2.15. □

2.17. In [39], a different proof of the cocompleteness of TTG is given, using a technique which is a generalization of the construction of a "canonical" extension of the action of a subgroup to an action of the whole group. It is also related to the construction of "induced representations". Both techniques are very important in the theory of actions of compact groups. See [9] and [40]. The methods, used in [39] also indicate how examples can be constructed which show that K does not preserve all coproducts.

3. SUBCATEGORIES OF TTG

3.1. Let A and B denote subcategories of TOPGRP and TOP, respectively, and set $X := K^{\leftarrow}[A \times B]$. Then X is a subcategory of TTG. The restrictions and corestrictions of the functors K, G and S will be denoted by the same symbols; so we have $K: X \to A \times B$, $G: X \to A$ and $S: X \to B$.

3.2. If one wants to show that $K: X \to A \times B$ is monadic using the same methods as in 2.3, one has to require, among others, that $G \times X$ is an object in B for every object (G,X) in $A \times B$. This condition appears to be rather harmless at first sight. However, a large portion of Topological Dynamics deals with actions of discrete groups on compact Hausdorff spaces (cf.[16]). So one might try to apply category theory in this field by taking B: = COMP (the category of all compact Hausdorff spaces) and A a category having discrete groups as objects. Then the above condition is only fulfilled if the objects of A are all finite. For Topological Dynamics, the restriction to finite groups is unacceptable. For other parts of the theory of ttg's, actions of finite groups on compact spaces is very important: it is one of the corner stones of the general theory of actions of compact Lie groups (cf.[29], p.222).

Although monadicity of the functor $K: X \to A \times B$ may be unattractive in view of practical purposes, K and X do have nice properties under rather mild conditions. The proofs of the following propositions can be found in [39], section 4.

3.3. PROPOSITION. *Suppose that the inclusion functor of* B *into* TOP *preserves all limits. Then the functor* $K: X \to A \times B$ *creates all limits. Hence, if* A *and* B *are complete, then so is* X, *and all limits and monomorphisms are preserved and reflected by* K. □

3.4. PROPOSITION. *Suppose* B *is a full subcategory of* TOP. *If either* $A \subseteq B$ *or* B *is productive and closed hereditary, then* K *preserves and reflects monomorphisms.* □

3.5. If one wants to show that X is cocomplete or that K: X → A × B preserves epimorphisms, then the proofs given for TTG cannot be adapted to the present situation, unless the restricted functor S_1: X → TOP actually sends X into B. So this brings us directly to the first problem, mentioned in section 1. In addition, something must be known about the epimorphisms in A.

Although the question of which nice properties of the phase space of a ttg are inherited by the orbit space (and under what circumstances!) is very interesting, the following proposition avoids this problem (for details, cf.[39], section 4).

3.6. PROPOSITION. *Suppose that the following conditions are fulfilled:*

(i) *Epimorphisms in* A *have dense ranges;*

(ii) B *is a full subcategory of* HAUS, *having a terminal object;*

(iii) *If* Y *is an object in* B *and* A *is a closed subset of* Y, *then* $Y \cup_A Y$ *is in* B.

Then the functor K: X → A × B *preserves and reflects epimorphisms.* □

3.7. The conditions (ii) and (iii) are rather mild. Yet the above proposition is of restricted applicability, because of condition (i). Indeed, it is still unknown to me whether the category HAUSGRP satisfies condition (i). It is known, however, that COMPGRP does! Of course, the preceding proposition can also be applied to the very important case (which we neglected untill now) that the category A has only one object, a fixed topological group G, and only one morphism, namely 1_G. In that case, the category X will be denoted B^G (the category of all G-spaces with phase space in B).

3.8. I want to make now a few remarks concerning reflective subcategories of TTG. I shall restrict myself to the question of the "preservation of reflections" by the functor K for only one particular case. In view of the following lemma, it is the functor S which causes difficulties.

3.9. LEMMA. *Suppose that* X: = $K^{\leftarrow}$[A × B] *is a reflective subcategory of* TTG *and that* B *has a final object. Then* A *is a reflective subcategory of* TOPGRP. *In addition, if* ⟨G,X,π⟩ *is an object in* TTG *and*

$$\langle\psi,f\rangle : \langle G,X,\pi\rangle \to \langle H,Y,\sigma\rangle$$

is its reflection into X*, then* ψ: G → H *is a reflection of* G *into* A.

PROOF. This is a consequence of the fact that under rather mild conditions a functor having a right adjoint (c.q. the functor G) preserves reflections. For a different proof, see [39], p. 133. □

3.10. It is not difficult to show that $K^{\leftarrow}$[COMPGRP × COMP] is a reflective subcategory of TTG. So by the preceding lemma, the reflection of an object $\langle G,X,\pi\rangle$ of TTG into this subcategory has the form

$$\langle\alpha_G,f\rangle : \langle G,X,\pi\rangle \to \langle G^c,Y,\sigma\rangle,$$

where α_G: G → G^c is the *Bohr compactification* of G. In general, f: X → Y is not the reflection of X into COMP. In fact, there are examples which show that Y can be a one-point space even if X is a non-trivial compact Hausdorff space. The problem whether Y is trivial or not has been important in Topological Dynamics, and is related to many interesting questions. This follows from the following observation (cf.[39], p.141):

Using the above notation with X *a compact Hausdorff space, the morphism*

$$\langle 1_G,f\rangle : \langle G,X,\pi\rangle \to \langle G,Y,\sigma^\alpha\rangle$$

where $\sigma^\alpha(t,y) := \sigma(\alpha(t),y)$, $(t,y) \in G \times Y$, *is just the maximal equicontinuous factor of* $\langle G,X,\pi\rangle$.

For definitions and results concerning maximal equicontinuous factors, cf. [17], [18] and [35].

3.11. Using the notation explained in 3.7, it is not difficult to show that $COMP^G$ is a reflective subcategory of TOP^G, for any topological group G. Let the reflection of the G-space $\langle G,X,\pi\rangle$ into $COMP^G$ be denoted by

$$\langle 1_G,k\rangle : \langle G,X,\pi\rangle \to \langle G,Z,\zeta\rangle.$$

If G is discrete, k: $X \to Z$ is just the reflection of X into COMP. There are examples which show that for non-discrete groups (e.g. $G=\mathbb{R}$) k: $X \to Z$ may be not the reflection of X into COMP. See [11]; it can also be shown that $\langle G,G,\lambda\rangle$ gives such an example, provided $RUC^*(G) \neq C^*(G)$. For the inequality $RUC^*(G) \neq C^*(G)$, cf. [13]. The reflection of $\langle G,G,\lambda\rangle$ into $COMP^G$ plays an important role in Topological Dynamics; there it is called the *greatest* (or *maximal*) G-*ambit*. See [10] and the references given there.

I do not know whether the mapping k: $X \to Z$ is a topological embedding if X is a Tychonoff space. I have some partial results, including the cases that X is locally compact Hausdorff or that $\langle G,X,\pi\rangle$ has the form $\langle G,(G/H)\times Y,\sigma\rangle$ with $\sigma^t(sH,y) := (tsH,y)$, $t, s \in G$, $y \in Y$, where H is a closed subgroup of G. See [37].

4. THE CATEGORY TTG_* AND GENERALIZATIONS

4.1. The objects of TTG_* are the same as the objects of TTG, viz. the ttg's. The categories differ from each other with respect to their morphisms. We shall first give a brief motivation for the definition of the morphisms in TTG_*. The idea stems from the following problem: Given a ttg $\langle G,X,\pi\rangle$, does there exist a ttg $\langle H,Y,\sigma\rangle$ such that

(i) Y is a topological vector space ;

(ii) Each σ^t, $t \in H$, is an invertible continuous linear operator on Y*);

(iii) X can be embedded in Y as an invariant subset in such a way that $\bar{\pi}[G] = \{\sigma^t|_X : t \in H\}$.

If $\langle G,X,\pi\rangle$ is effective (i.e. $\pi^t \neq \pi^s$ if $t \neq s$), then it follows from (iii) that we obtain a homomorphism ψ: $H \to G$ such that for every $t \in H$ the following diagram commutes

$$\begin{array}{ccccc} X & \xrightarrow{\pi^{\psi(t)}} & X & \qquad & G \\ f\downarrow & & \downarrow f & & \uparrow\psi \\ Y & \xrightarrow{\sigma^t} & Y & & H \end{array}$$

*) Such a ttg $\langle H,Y,\sigma\rangle$ will be called *a linear ttg*.

Here f: $X \to Y$ is the embedding mapping of X into Y.

For more details about the above mentioned linearization problem we refer to section 5. At this point we are only interested in the diagram which expresses the relationship between ψ and f. We shall use it in the following definition:

4.2. The object class of TTG_* is the class of all ttg's. A *morphism in* TTG_* *from* $\langle G,X,\pi\rangle$ *to* $\langle H,Y,\sigma\rangle$ is a morphism (ψ^{op},f): $(G,X) \to (H,Y)$ in the category $TOPGRP^{op} \times TOP$ such that for every $t \in H$ the diagram in 4.1 commutes (now f is not necessarily an embedding). *Notation:* $\langle\psi^{op},f\rangle$: $\langle G,X,\pi\rangle \to \langle H,Y,\sigma\rangle$. The composition of the morphisms in TTG_* is defined coordinatewise.

4.3. The obvious forgetful functor from TTG_* to $TOPGRP^{op} \times TOP$ will be denoted by K_*. *It can be shown that this functor preserves all colimits.* Using this, it is fairly easy to construct an example which shows that *the category* TTG_* *is not complete* (the example is related to the one in 2.13). It can also be shown that K_* *preserves all monomorphisms.* In particular it follows that TTG_* is well-powered. We shall see in 4.7 below that TTG_* has a coseparator. According to Theorem 23.14 in [24], a complete, well-powered category having a coseparator is cocomplete. *It follows, that* TTG_* *is not complete*.

4.4. We want to say something more about the existence of coseparators in TTG_*, also because such objects are related to the general embedding problem, mentioned in 4.1. The notation will be similar to the notation in section 3, except for some obvious modifications. Thus, A and B are subcategories of TOPGRP and TOP, respectively, and $X_* := \overleftarrow{K_*}[A^{op} \times B]$. Moreover, for any object (G,X) in $TOPGRP^{op} \times TOP$, the evaluation mapping $f \mapsto f(e)$: $C_c(G,X) \to X$ will be denoted by δ_X^G.

4.5. LEMMA. *For every object* (G,X) *in* $TOPGRP^{op} \times TOP$ *with* G *a locally compact Hausdorff group, the pair* $(\langle G,C_c(G,X),\tilde{\rho}\rangle, (1_G^{op},\delta_X^G))$ *is a co-universal arrow for* (G,X) *with respect to the functor* K_*.

PROOF. Consider the following diagram:

$$\begin{array}{ccc} \langle G, C_c(G,X), \tilde{\rho}\rangle & \qquad & (G, C_c(G,X)) \xrightarrow{(1_G^{op}, \delta_G^X)} (G,X) \\ \uparrow \langle \psi^{op}, f\circ\underline{\sigma}(-)\circ\psi\rangle & & (\psi^{op}, f\circ\underline{\sigma}(-)\circ\psi) \uparrow \quad \nearrow (\psi^{op}, f) \\ \langle H, Y, \sigma\rangle & & (H,Y) \end{array}$$

Here for every $y \in Y$, $\underline{\sigma}(y) := \sigma_y : H \to Y$, so that $f\circ\underline{\sigma}(-)\circ\psi : y \mapsto f\circ\sigma_y\circ\psi : G \to X$. Observe, that local compactness of G is needed to ensure that $\tilde{\rho}: G \times C_c(G,X) \to C_c(G,X)$ is continuous. □

4.6. COROLLARY. *If* A *has a generator* G *which is locally compact Hausdorff and if* B *has a coseparator* X *such that* $C_c(G,X)$ *is an object in* B, *then* $\langle G, C_c(G,X), \tilde{\rho}\rangle$ *is a coseparator in* X_*.

PROOF. Apply the dual of Prop. 31.11 of [24]. □

4.7. EXAMPLES.

(i) Let E be the indiscrete 2-point space. Then the ttg $\langle \mathbb{Z}, E^{\mathbb{Z}}, \tilde{\rho}\rangle$ is a coseparator in TTG_*.

(ii) $\langle \mathbb{Z}, [0,1]^{\mathbb{Z}}, \tilde{\rho}\rangle$ is a coseparator for the full subcategories of TTG_*, determined by all ttg's with a Tychonoff, resp. with a compact T_2, phase space.

(iii) If G is a fixed locally compact Hausdorff group, then $\langle G, C_c(G,[0,1], \tilde{\rho}\rangle$ is a coseparator for the full subcategory of TOP^G defined by all Tychonoff G-spaces (not for $COMP^G$, unless G is discrete).

4.8. In the remainder of this section, A shall denote the full subcategory of TOPGRP, defined by all *locally compact Hausdorff groups*, and $X_* := K_*^{\leftarrow}[A^{op} \times TOP]$ (so we take $B := TOP$). In this case, it follows immediately from 4.5, that the functor $K_*: X_* \to A^{op} \times TOP$ has a right adjoint. This follows also from our next theorem:

4.9. THEOREM. *The functor* $K_*\colon X_* \to A^{op} \times \mathrm{TOP}$ *is comonadic. Consequently,* X_* *is a finitely cocomplete (for so is* $A^{op} \times \mathrm{TOP}$*), and all existing colimits and all epimorphisms are preserved and reflected by* K_*.

PROOF. Details will be published elsewhere. We only mention that in the proof essential use is made of the canonical homeomorphisms

$$C_c(G\times G,X) \simeq C_c(G,C_c(G,X)),$$

$$C_c(G\times X,X) \simeq C_c(X,C_c(G,X)),$$

G any locally compact Hausdorff space and X an arbitrary topological space. □

4.10. If G is a fixed locally compact Hausdorff topological group, then the category TOP^G can be considered as a subcategory of TTG_*. Similar to 4.9, one shows that TOP^G may be considered as a category of coalgebras over a comonad in TOP. On the other hand, similar to 2.3, TOP^G may be considered as a category of algebras over a monad in TOP. The monad and the comonad considered here are nicely related: it can be shown that they are adjoint to each other according to the definition given in [15]. Although this seems to be known, I could find no references to this fact in the literature.

4.11. One might conclude from the above remarks that ttg's with locally compact phase groups are the nice objects which deserve further study. Although this conclusion is true as far as it concerns the applications, from a categorical point of view there is a much nicer class of objects. Indeed, the homeomorphisms used in the proof of 4.9 are an indication of the fact that we should work in the cartesian closed category KR of all k-spaces. The proper objects are the systems $[G,X,\pi]$ where G is a k-group (i.e. a group G with a k-topology making the mapping $(s,t) \mapsto st^{-1}\colon G \otimes G \to G$ continuous), X is a k-space and $\pi\colon G \otimes G \to X$ is a continuous mapping satisfying the usual equations (here $\otimes$ denotes the product in the category KR: the k-refinement of the cartesian product). With the class of these k-ttg's as object class, one can form the categories k-TTG and k-TTG_* (morphisms similar to TTG and TTG_*, respectively). The study of these categories is initiated in [39].

5. LINEARIZATION OF ACTIONS

5.1. The general problem which we described in 4.1 has a trivial solution if we work in the category TTG_*. Indeed, lemma 5.2 below shows that the only condition which must be imposed on a ttg $\langle G,X,\pi\rangle$ in order that the action can be linearized, is that X can be embedded in a topological vector space. If we restrict ourselves to Hausdorff topological vector spaces, this means exactly that X is a Tychonoff space. In the following lemma, G_d is the group G with the discrete topology, and $\iota: G_d \to G$ is the identity mapping.

5.2. LEMMA. *If* $\langle G,X,\pi\rangle$ *is a ttg with* X *a Tychonoff space then there exists a morphism* $\langle \iota^{op},f\rangle: \langle G,X,\pi\rangle \to \langle G_d,V,\sigma\rangle$ *in* TTG_* *such that* V *is a topological vector space,* σ *is a linear action, and* $f: X \to V$ *is a topological embedding.*

PROOF. There exists a topological embedding $g: X \to \mathbb{R}^\kappa$ for some cardinal number κ. Moreover, the mapping $\underline{\pi}: x \mapsto \pi_x: X \to C_c(G,X)$ is a topological embedding. Hence $f: = C_c(G,g)\circ\underline{\pi}$ is a topological embedding of X into the topological vector space $C_c(G,\mathbb{R}^\kappa) =: V$. Plainly, $\langle G_d,C_c(G,\mathbb{R}^\kappa),\tilde{\rho}\rangle$ is a ttg, and $\langle \iota^{op},f\rangle$ is a morphism in TTG_*. □

5.3. In order to make the problem more interesting, the following extra conditions will be imposed. First, a G-space $\langle G,X,\pi\rangle$ should be linearized in a G-space rather than in a G_d-space. Second, if X is a metric space, then $\langle G,X,\pi\rangle$ should be linearized in a *Hilbert* G-*space*. And finally, a large class of G-spaces should be linearized simultaneously in one and the same linear G-space.

The proof of the following theorem is a modification of the proof of lemma 5.2 above. Observe, that the apparent relationship with lemma 4.5 has a categorical background (cf. [24], Prop. 19.6).

5.4. THEOREM. *Let* G *be a locally compact Hausdorff group and let* κ *be a cardinal number. Then every* G-*space* $\langle G,X,\pi\rangle$ *with* X *a Tychonoff space of weight* $\leq \kappa$ *can equivariantly be embedded in the linear* G-*space* $\langle G,C_c(G,\mathbb{R})^\kappa, \tilde{\rho}^\kappa\rangle$. □

Using similar ideas, in combination with results from [5] and [32], the following theorem can be proved. Here $H(\kappa)$ is the Hilbert sum of κ copies of the Hilbert space $L^2(G)$, and the action $\sigma(\kappa)$ induces on each copy of $L^2(G)$ a "weighted" right translation. For a proof, see [36].

5.5. THEOREM. *Let* G *be a sigma-compact locally compact Hausdorff group and let* κ *be a cardinal number. Then there exists a linear* G-*space* $\langle G,H(\kappa),\sigma(\kappa)\rangle$ *in which every* G-*space* $\langle G,X,\pi\rangle$ *with* X *a metric space of weight* $\leq \kappa$ *can be equivariantly embedded.* □

5.6. Let G be as in 5.5. and assume, for convenience, that G is infinite. The weight of the Hilbert space $H(\kappa)$ equals $\kappa \cdot w(G)$, where $w(G)$ is the weight of G (for compact groups this is well-known; the proof for the non-compact case can be found in [36]). If we take $\kappa = w(G)$, then $H(\kappa)$ is isomorphic to $L^2(G)$. So there is an action σ of G on $L^2(G)$ such that $\langle G,L^2(G),\sigma\rangle$ is a linear ttg in which every metric G-space $\langle G,X,\pi\rangle$ with $w(X) \leq w(G)$ can be equivariantly embedded. No explicit description of σ can be given in this case. However, there is an action τ of G on $L^2(G\times G)$ which can easily be described explicitly, such that $\langle G,L^2(G\times G),\tau\rangle$ is a linear ttg in which every metric G-space $\langle G,X,\pi\rangle$ can equivariantly be embedded, provided $w(X) \leq L(G)$, the Lindelöf degree of G). The proof is highly non-categorical; see [38].

5.7. One of the most notable early results on linearizations of actions is BEBUTOV's theorem. See [26] and also [23]. These results have applications in the theory of differential equations. Also related to differential equations are the results in [12]. In these three papers, only actions of $\mathbb{R}$ are linearized. Actions of Lie groups are considered, among others, in [33]. See also [34] and the references given there. Linearizations in Hilbert spaces of actions of more general locally compact groups appear in [3] and [5], where earlier work of COPELAND and DE GROOT ([14],[20]) was generalized. More information about the history of these results can be found in [3] and in [6].

REFERENCES.

[1] ABRAHAM, R. & J.E. MARSDEN, *Foundations of mechanics*, Benjamin, New York, 1967.

[2] ABRAHAM, R. & J. ROBBIN, *Transversal mappings and flows*, Benjamin, New York, 1967.

[3] BAAYEN, P.C., *Universal morphisms*, Mathematical Centre Tracts no. 9, Mathematisch Centrum, Amsterdam, 1964.

[4] BAAYEN, P.C., Topological linearization of locally compact transformation groups, *Report no. 2, Wiskundig Seminarium, Vrije Universiteit*, Amsterdam, 1967.

[5] BAAYEN, P.C. & J. DE GROOT, Linearization of locally compact transformation groups in Hilbert space, *Math. Systems Theory* 2(1968), 363-379.

[6] BAAYEN, P.C. & M.A. MAURICE, Johannes de Groot 1914-1972, *General Topology and Appl.* 3(1973), 3-32.

[7] BECK, A., *Continuous flows in the plane*, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[8] BHATIA, N.P. & G.P. SZEGÖ, *Stability theory of dynamical systems*, Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[9] BREDON, G.E., *Introduction to compact transformation groups*, Academic Press, New York, 1972.

[10] BROOK, R.B., A construction of the greatest ambit, *Math. Systems Theory* 4(1970), 243-248.

[11] CARLSON, D.H., Extensions of dynamical systems via prolongations, *Funkcial. Ekvac.* 14(1971), 35-46.

[12] CARLSON, D.H., Universal dynamical systems, *Math. Systems Theory* 6(1972), 90-95.

[13] COMFORT W.W. & K.A. ROSS, Pseudocompactness and uniform continuity in topological groups, *Pacific J. Math.* 16(1966), 483-496.

[14] COPELAND Jr., A.H. & J. DE GROOT, Linearization of a homeomorphism, *Math. Annalen* 144 (1961), 80-92.

[15] EILENBERG, S. & J.C. MOORE, Adjoint functors and triples, *Illinois J. Math.* 9(1965), 381-398.

[16] ELLIS, R., *Lectures on topological dynamics*, Benjamin, New York, 1969.

[17] ELLIS, R. & W.H. GOTTSCHALK, Homomorphisms of transformation groups, *Trans. Amer. Math. Soc.* 94(1960), 258-271.

[18] ELLIS, R. & H. KEYNES, A characterization of the equicontinuous structure relation, *Trans. Amer. Math. Soc.* 161(1971), 171-183.

[19] GOTTSCHALK, W.H. & G.A. HEDLUND, *Topological dynamics*, Amer. Math. Soc. Colloquium Publications, Vol. 36, Providence, R.I., 1955.

[20] GROOT, J. DE, Linearization of mappings, in *General topology and its relation to modern analysis and algebra*, Proc. 1961 Prague Symposium, Prague, 1972, p.191-193.

[21] HAJEK, O., *Dynamical systems in the plane*, Academic Press, New York, 1968.

[22] HAJEK, O., Parallelizability revisited, *Proc. Amer. Math. Soc.* 27(1971), 77-84.

[23] HAJEK, O., Representations of dynamical systems, *Funkcial. Ekvac.* 14(1971), 25-34.

[24] HERRLICH, H. & G.E. STRECKER, *Category theory*, Allyn and Bacon Inc., Boston, 1973.

[25] HIRSCH, M.W. & S. SMALE, *Differential equations, dynamical systems, and linear algebra*, Academic Press, New York, 1974.

[26] KAKUTANI, S., *A proof of Bebutov's theorem*, J. Differential Equations 4(1968), 194-201.

[27] KOSZUL, J.L., *Lectures on groups of transformations*, Tata Institute of Fundamental Research, Bombay, 1965.

[28] MACLANE, S., *Categories for the working mathematician*, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

[29] MONTGOMERY, D. & L. ZIPPIN, *Topological transformation groups*, Interscience, New York, 1955.

[30] NEMYCKIĬ, V.V., Topological problems in the theory of dynamical systems, *Uspehi Mat. Nauk.* 4(1949), no. 6(34), 91-153 (English translation in: AMS Translation Series 1, Vol. 5, p.414-497).

[31] NEMYCKIĬ, V.V. & V.V. STEPANOV, *Qualitative theory of differential equations*, Princeton University Press, Princeton, N.J., 1960.

[32] PAALMAN - DE MIRANDA, A.B., A note on W-groups, *Math. Systems Theory* 5(1971), 168-171.

[33] PALAIS, R.S., Slices and equivariant embeddings, in: A. BOREL et al., *Seminar on transformation groups*, Annals of Mathematics Studies 46, Princeton University Press, Princeton, N.J., 1960, p.101-115.

[34] PALAIS, R.S., On the existence of slices for actions of non-compact Lie groups, *Ann. of Math.* 73(1961), 295-323.

[35] PELEG, R., Weak disjointness of transformation groups, *Proc. Amer. Math. Soc.* 33(1972), 165-170.

[36] VRIES, J. DE, A note on topological linearization of locally compact transformation groups in Hilbert space, *Math. Systems Theory* 6(1972), 49-59.

[37] VRIES, J. DE, Can every Tychonoff G-space equivariantly be embedded in a compact Hausdorff G-space? *Math. Centrum, Amsterdam, Afd. Zuivere Wisk.*, ZW 36, 1975.

[38] VRIES, J. DE, A universal topological transformation group in $L^2(G\times G)$, *Math. Systems Theory* 9(1975), 46-50.

[39] VRIES, J. DE, *Topological transformation groups I (a categorical approach)*, Mathematical Centre Tracts, no. 65, Mathematisch Centrum, Amsterdam, 1975.

[40] WALLACH, N.R., *Harmonic analysis on homogeneous spaces*, Marcel Dekker, Inc., New York, 1973.